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Report

WaveKin Theory Manual

Theoretical background for the wave kinematics module WaveKin, developed at MARINTEK.

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WaveKin

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ABSTRACT
WaveKin is a program module intended for use by MARINTEK programs such as Riflex and Simo. It provides wave kinematics based on the theory of 2nd order Stokes waves in finite water depth, assuming a long-crested sea-state. This document describes the basic theory behind the kinematics calculations.

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Contents

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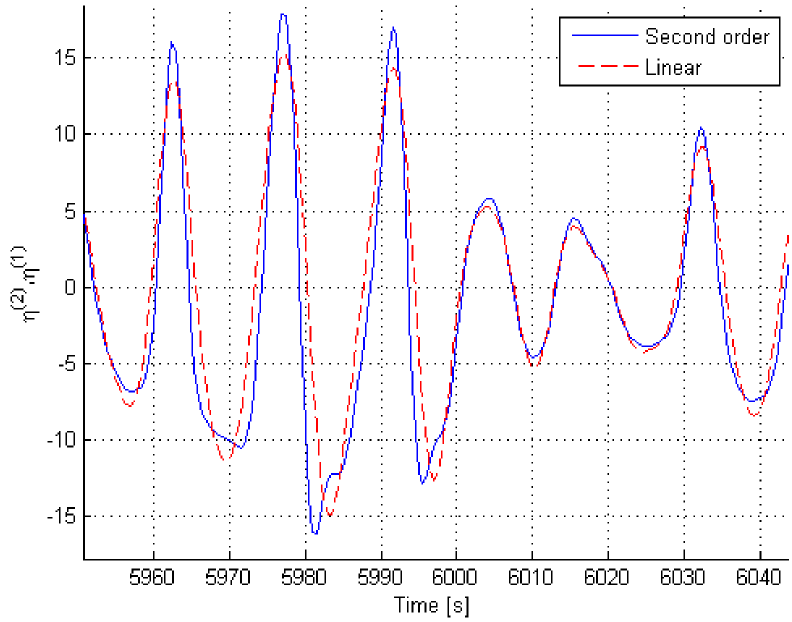

Figure 1: Wave elevation, with and without the second order contribution.

`WaveKin` is a program module used by the `MARINTEK` programs `Riflex` and `Simo`. It provides wave kinematics based on the theory of 2nd order Stokes waves in finite water depth.

1 Introduction

Input to `WaveKin` is the first order wave elevation, together with water depth and the depth co-ordinate at which kinematics are to be calculated. A second order, non-linear contribution to the wave elevation is calculated and added to the first order elevation. Fluid velocities and accelerations are calculated from the velocity potential, up to and including second order.

An impression of the effect of including second order terms is given in Fig. 1. For moderate elevations the second order contribution is insubstantial. For large elevations we obtain a non-linear behaviour which is seen as a flattening of troughs and sharpening of crests [1].

Currently, second order wave kinematics are available for long-crested sea-states only.

2 Stokes theory

In this section, Stokes theory is presented. First, the governing set of differential equations are presented. Then, it is shown how a Taylor expansion of the velocity potential around the mean surface level leads to a series expansion in terms of wave steepness. Finally, analytical solutions of the velocity potential and wave elevation are given up to and including second order.

[1] The wave shown is also discussed in Carl Trygve Stansberg *etal*, "Second-Order Random Wave Kinematics and Resulting Loads on a Bottom-Fixed Slender Monopile." ASME 2013 32nd International Conference on Ocean, Offshore and Arctic Engineering. American Society of Mechanical Engineers, 2013.

2.1 The wave potential

Stokes theory of ocean waves assumes incompressible, irrotational flow, described by a scalar velocity potential $\Phi(\vec{x}, t)$ which satisfies

$$\nabla^2\Phi = 0, \tag{1}$$

with velocities u given by spatial partial derivatives

$$\vec{u} = \nabla\Phi. \tag{2}$$

2.2 Boundary conditions

Assuming an impermeable sea-floor of constant mean depth h, we have $u_z(z = -h) = 0$, which we write in terms of the wave potential as

$$\partial_z\Phi = 0 \quad \text{at} \;\; z = -h. \tag{3}$$

The free surface of the wave is given by $z = \eta(x, y, t)$. For long-crested sea states the y co-ordinate is irrelevant, and ignored in the following.

Two boundary conditions are imposed at the wave surface. The kinematic boundary condition

$$\partial_t\eta + \partial_x\Phi\partial_x\eta + \partial_y\Phi\partial_y\eta - \partial_z\Phi = 0 \quad \text{at} \;\; z = \eta, \tag{4}$$

which states that a particle at the surface moves together with the surface, and the dynamic boundary condition

$$\partial_t\Phi + \frac{1}{2}\left|\vec{u}\right|^2 + g\eta = 0 \quad \text{at} \;\; z = \eta. \tag{5}$$

which follows from assuming constant pressure along the surface. Here, g is the acceleration of gravity.

Together with Eq. 1, which applies within the bulk of the ocean, Eqs. 3, 4 and 5 completely specify the set of differential equations from which Φ is obtained.

2.3 Series expansion of the surface elevation

In order to obtain analytic results for Φ, a perturbative approach is taken.

The velocity potential at the wave surface is extrapolated onto the mean surface level $z = 0$ using a Taylor expansion, with terms of higher order in the expansion expected to contribute at higher order in the ordering parameter ϵ.

The expansion is, to order $\mathcal{O}(\epsilon^2)$,

$$\Phi|_{z>0} = \Phi|_{z=0} + z\left\{\partial_z\Phi\right\}_{z=0} + \frac{1}{2}z^2\left\{\partial_z^2\Phi\right\}_{z=0} + \dots. \tag{6}$$

Likewise, the surface elevation is written as a series expansion,

$$\eta = \eta^{(1)} + \eta^{(2)} + \dots, \tag{7}$$

where it is usually assumed that $\eta^{(1)}$ is a *linear* wave. This is a construction which allows us to study the non-linear effects which appear at second order and beyond.

The surface elevation is a function of x and t, which in Fourier space is wavenumber k and angular frequency ω. They are related through the dispersion relation for surface waves at finite water-depth,

$$\omega^2 = gk\tanh kh, \tag{8}$$

which is complete to second order. At third order, the dispersion relation includes an additional term.

The first order contribution $\eta^{(1)}$ is provided to `WaveKin` by its Fourier components a_i,

$$\eta^{(1)} = \sum_{\mathrm{i}=1}^{N} \frac{1}{2} a_\mathrm{i} e^{i\psi_\mathrm{i}}, \tag{9}$$

where $\psi_\mathrm{i} = \omega_\mathrm{i} t - k_\mathrm{i} x - \varepsilon_\mathrm{i}$ is the phase function, with angular frequency $\omega_\mathrm{i} = i\Delta\omega$, wavenumber $k_\mathrm{i} = k(\omega_\mathrm{i})$ and phase ε_i. For a simulated wave from a given spectrum, the phase will typically be drawn from a uniform probability distribution.

The sum of Eq. 9 runs over positive frequencies only. Becuase $\eta(x,t)$ is real-valued, $a_{-\mathrm{i}} = \bar{a}_\mathrm{i}$. Equivalent relations exist for the time-series of velocities and accelerations; `WaveKin` therefore works with one-sided spectra only. Time-series may have a non-zero mean value, therfore the one-sided spectra include the zero-frequency components. The zero-frequency component for surface elevation a_0 is also included in the one-sided spectrum, though its value should be zero or near zero.

The expansion is in terms of an ordering parameter ϵ. It can be seen that higher orders in the expansions of the boundary conditions at the wave surface results in terms with higher powers of spatial derivatives, surface elevation and the velocity potential. In Fourier space, spatial derivatives result in additional factors of k, and we have constructed the first order surface elevation to be a sum of harmonics with amplitude $|a|$.

The combination of factors of k and a provides a physical meaning to the ordering parameter. The wave steepness is proportional to ak, for a harmonic wave elevation $\eta_k = a \sin kx$. In order for the series expansion to converge, the linear wave should have a spectral content which satisfies $|a_\mathrm{i}|k_\mathrm{i} < \epsilon$ for all i.

In addition to the Taylor series expansion of the wave surface boundary conditions and the expansion of η, the velocity potential is written as a series expansion

$$\Phi = \Phi^{(1)} + \Phi^{(2)} + \dots. \tag{10}$$

A solution is obtained for each term of the expansion by ordering the terms in the wave surface boundary conditions, setting $z = \eta$ in Eq. 6. Terms at equal order in ϵ are solved for independently.

Solving for the first order velocity potential gives

$$\Phi^{(1)}(x,z,t) = \sum_{\mathrm{i}=-N}^{N} \frac{1}{2} \frac{iga_\mathrm{i}}{\omega_\mathrm{i}} \frac{\cosh k_\mathrm{i}(z+h)}{\cosh k_\mathrm{i} h} e^{i\psi_\mathrm{i}}. \tag{11}$$

2.4 Second order solution

Following Marthinsen and Winterstein (1992) [2], the velocity potential is given to second order by a rather complex expression. It is analytic, and therefore supports differentiation, allowing the calculation of velocities and accelerations.

The second order contribution to surface elevation is given by a similarly complex expression.

The solutions for both $\Phi^{(2)}$ and $\eta^{(2)}$ depend on products of Fourier components which contribute to the sum of the respective frequencies (wavenumbers). Because the double-sided spectrum can be expressed as a single-sided spectrum, the second order contributions are naturally separated into sum-frequency and difference-frequency contributions.

Because the sum- and difference-frequency components only depend on the sum and difference of frequencies (wavenumbers) and the Fourier components satisfy $a_{-\mathrm{i}} = \bar{a}_\mathrm{i}$, the calculations of second order contributions to surface elevation, velocities and accelerations can be considerably simplified. The efficient calculation of these quantities is a significant feature of `WaveKin`.

The difference-frequency terms give finite contributions to the zero-frequency component of the second order elevation and velocity potential. This corresponds to finite mean-values for η and u_x. This can optionally be avoided by setting all zero-frequency components to zero.

[2] Tom Marthinsen and Steven R. Winterstein. "On the skewness of random surface waves." The Second International Offshore and Polar Engineering Conference. International Society of Offshore and Polar Engineers, 1992.

3 Kinematics

Velocities and accelerations are obtained from applying partial derivatives to the wave potential. The acceleration to second order is given by

$$\vec{a}^{(1+2)} = \frac{\mathrm{d}}{\mathrm{dt}}(\vec{u}^{(1)} + \vec{u}^{(2)}) \approx \frac{\partial}{\partial t}(\vec{u}^{(1)} + \vec{u}^{(2)}). \tag{12}$$

In the above expression, convective terms are neglected. In the current implementation of `WaveKin`, the inclusion of these terms is optional, with the default option being to ignore them. This reflects the general assumption and established convention that convection terms are small and can be neglected.

Differentiation is performed in Fourier space.

3.1 Convection terms

Convection terms can optionally be included in the accelerations returned by `WaveKin`. These are calculated by taking spatial derivatives of the velocity field (in the frequency domain) and multiplying with the velocity field (in the time domain).

3.2 Kinematics above the mean surface level

The velocity potential $\Phi(x, z, t)$ is extrapolated above $z = 0$ according to the Taylor expansion of Eq. 6. Kinematics are obtained from the extrapolated potential by spatial and temporal differentiation in the same manner as for the velocity potential at and below $z = 0$. However, it should be noted that the second order contribution at $z > 0$ involves the first order velocity potential $\Phi^{(1)}$. There are no quadratic transfer functions involved above mean surface level until third order contributions.

4 Validation

The calculation of elevation and kinematics is compared with previous calculations, using the 4hr wave elevation record of Stansberg *etal* (2013).

Figs. 2, 3 and 4 show time series of surface elevation, horizontal and vertical velocities, horizontal and vertical accelerations and power spectra for the wave studied in Stansberg *etal* (2013). Kinematics are calculated at $z = -2$m. When the surface elevation drops beneath this depth the node is considered to be ”dry” and no results are plotted.

The power spectra (shown with no filtering or smoothing) are the total spectrum and that of the second order contribution only, showing that the second order contribution separates into low-frequency (difference-frequency) and high-frequency (sum-frequency) contributions.

The `WaveKin` implementation has been compared with the implementation used by Stansberg *etal* (2013). Results are found to be in agreement.

4.1 Depth profile

Fig. 5 shows velocities and accelerations obtained from an irregular wave generated from a JONSWAP spectrum with $H_s = 7.15$m, $T_p = 12.4$s and $\gamma = 1.52$. The velocities and accelerations are plotted on the horizontal axis with the depth co-ordinate on the vertical axis.

When the depth profile is animated, a swaying motion is observed. Amplitudes are largest near the surface, as expected. Above the surface there are no kinematics. For the purpose of extrapolation between a ”wet” and a ”dry” node on a partially submerged structure, a useful convention is to keep a constant value for kinematics at $z > \eta$ which is equal to the values at $z = \eta$.

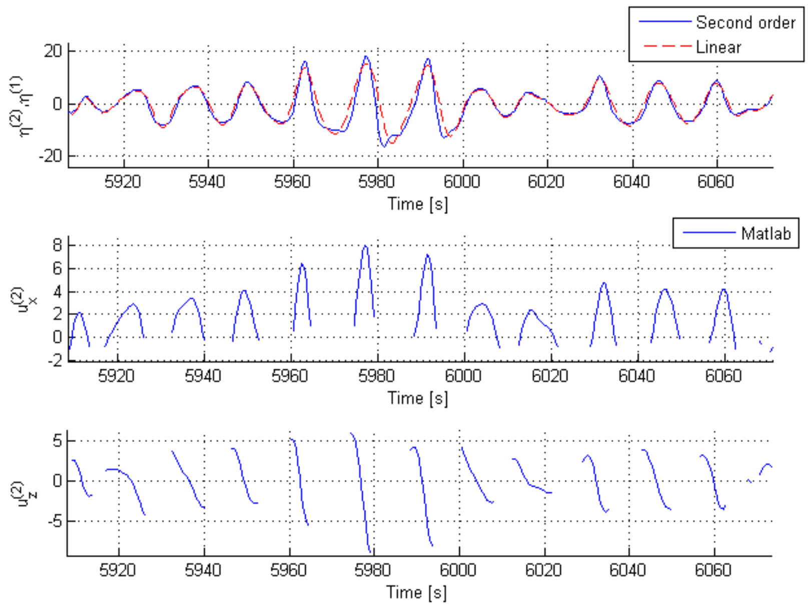

Figure 2: Time series of surface elevation, horizontal and vertical velocities. The surface elevation shows both the linear (1st order) and non-linear (1st and 2nd order) wave. The velocities are given at $z = -2$m. No results are plotted when the surface is below the chosen value of z.

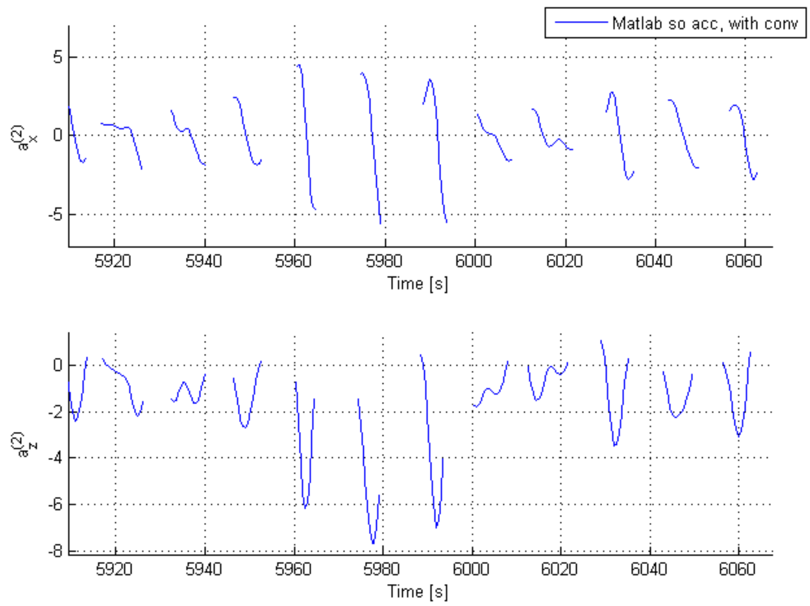

Figure 3: Time series of horizontal and vertical accelerations. Otherwise identical with Fig. 2. Includes convection terms.

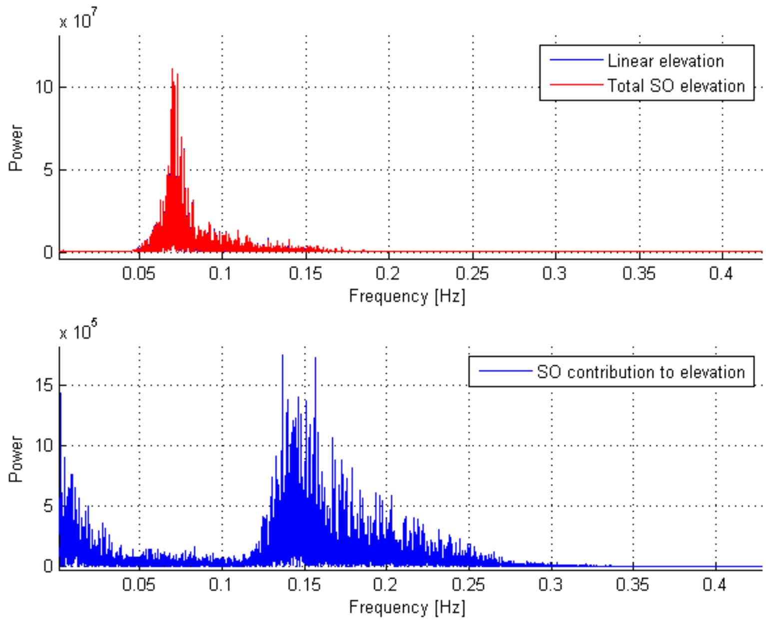

Figure 4: Power spectra of surface elevation, total (top) and second order contributions (bottom). Both spectra are for the surface elevation shown in Fig. 2.

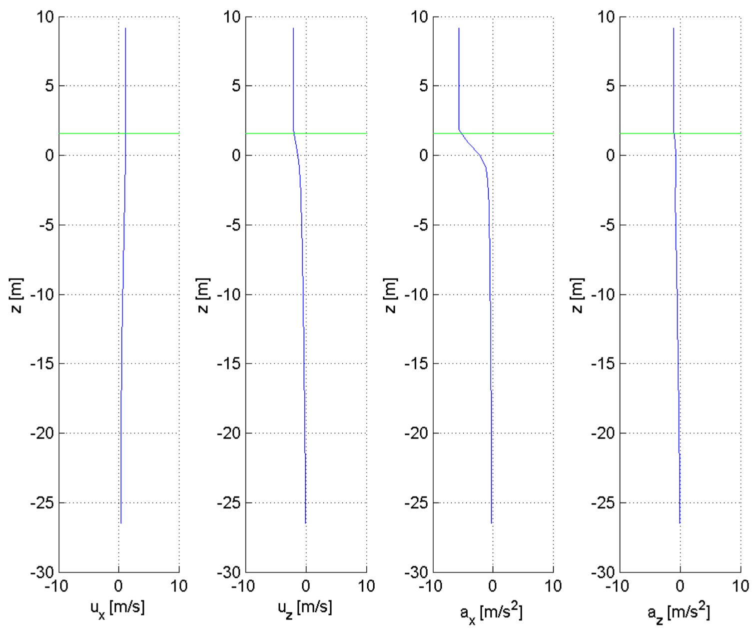

Figure 5: Depth profile of wave kinematics: horizontal and vertical velocities and horizontal and vertical accelerations, respectively. The green horizontal line marks the surface elevation.

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